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Aristotle on Geometrical Objects

by Ian Mueller (Chicago)

From the perspective of ontology or of epistemology the question may be asked: 'What is the nature of mathematical objects?' To ask the question ontologically amounts to asking for the real subjects, the things in the world, with which mathematics deals. Epistemologically the question is more likely to be directed at mathematical reasoning: 'What is mathematical reasoning about?'

Plato seems to have given the same answer to the question from both perspectives. It is even possible that his answer to the ontological question was inferred from his epistemological analysis of mathematics in some way like the following: Mathematicians reason as if they were dealing with objects that are different from all sensible things, perfectly fulfill given conditions, and are apprehensible by pure thought; mathematics is correct; therefore, there are such objects. Argument of this kind is also characteristic of the modern mathematical Platonist. For example, K. Gödel writes: "It seems to me that the assumption of such objects [classes and concepts] is quite as legitimate as the assumption of physical bodies, and there is quite as much reason to believe in their existence. They are in the same sense necessary to obtain a satisfactory system of mathematics as physical bodies are necessary for a satisfactory theory of our sense perceptions . . .[1]" Ancient and modern mathematical Platonism rests its case largely on being a direct inference from the nature of mathematical reasoning.

Other philosophers, however, begin their inquiries either by denying or being skeptical about the entities postulated by Platonists. The intuitionist in A. Heyting's "Disputation" is made to say, "We have no objection against a mathematician privately admitting any metaphysical theory he likes, but Brouwer's program entails that we study mathematics as something simpler, more immediate than metaphysics[2]." A consequence of this metaphysical

[1] K. Gödel, "Russell's Mathematical Logic," reprinted in *Philosophy of Mathematics: Selected Readings*, ed. P. Benacerraf and H. Putnam (Englewood Cliffs, N. J., 1964), p. 220.

[2] A. Heyting, *Intuitionism: An Introduction*, 2nd ed. (Amsterdam, 1966), p. 2; reprinted in *Philosophy of Mathematics*, p. 56.

or ontological skepticism is the intuitionist's denial of the reality of mathematical objects; they are "mental constructions".

Aristotle begins his philosophizing about mathematics with an ontology that excludes mathematical objects of the kind envisaged by Plato. For Aristotle the paradigms of real things are sensible substances like animals, plants, and the heavenly bodies. (I ignore the prime mover [or movers] on the ground that in the philosophy of Aristotle this is a "special case," being both fully real and completely abstract.) On the other hand, Aristotle does accept Plato's mathematical epistemology: mathematicians treat objects which are different from all sensible things, perfectly fulfill given conditions, and are apprehensible by pure thought. To resolve the discrepancy between his mathematical epistemology and his ontology, Aristotle is not willing to construe mathematical objects as merely mental constructions dependent on human thought for their existence. Aristotle, of course, does place emphasis on the role of human thinking in mathematics but he also accepts the Platonic assumption that there must be a significant correlation between the apparent objects of mathematical reasoning and the real world—the assumption, as Zeller puts it, "that the truth of knowledge keeps pace with the actuality of its object[3]." But for Aristotle this assumption precludes the merely mental existence of mathematical objects. In this paper I shall describe the way in which Aristotle attempts to develop an account of mathematical objects while leaving intact the beliefs just described.

I begin with a quotation from *Metaphysics* M.3 in which there occurs what looks like a very straightforward account:

> Just as the universal branches of mathematics are not about separate things apart from magnitudes and numbers but rather about these, although not as having magnitude or being divisible; obviously there could be assertions and proofs about sensible magnitudes, not as sensible but as of some sort. And just as there are many assertions about things simply as changing separate from the being (τί ἐστί) of all such things and their properties, but it is not thereby necessary that there be something changing separate from sensible things nor some distinct entity in them; so also there may be assertions and theories regarding changing

[3] E. Zeller, *Aristotle and the Earlier Peripatetics*, tr. B. Costelloe and J. Muirhead (London, 1897), p. 339.

> things, not as changing but only as bodies, or again only as planes, or only as lengths, or only as divisible, or as indivisible and having position, or only as indivisible . . . If geometry happens to be of sensibles but not as sensible, the mathematical sciences will not therefore be of sensibles nor of other things separate from these. Many properties belong essentially to things as having such and such characteristics, since there are properties peculiar to an animal as male or as female and yet there is no male or female separate from animals. Thus things may have properties just as lines or as planes . . .
>
> Each thing is investigated best if someone posits as separate what is not separate, which is what the arithmetician and the geometer do. For a man as man is one and indivisible; the arithmetician posited one indivisible thing and then inquired whether anything belonged to the man as indivisible. The geometer treats of things neither as man nor as indivisible but as solid[4].

The geometer, then, deals with sensible substances, not as sensible substances, but as solids, planes, lines, and points. But what does this mean? The mathematician never really reasons about a man, whether as solid, indivisible, or anything else. Sensible substances, moreover, no matter how they are treated, do not fulfill the idealized conditions imposed on mathematical objects. This Aristotle observes with regard to the question, "With what sort of things must the mathematician be supposed to deal? Certainly not with the things around us. For none of these is like what the mathematical sciences investigate[5]." A bronze sphere (even as solid) does not touch an iron bar (even as plane or line) in a point.

Formulas like 'sensible substances as solids' do not adequately represent Aristotle's account of mathematical objects. *Metaphysics* M.3 is designed primarily to provide an alternative to the claim that there are actually existing mathematical objects either separate from or in sensible substances. M.3 reaffirms Aristotle's ontology and makes it seem closer to his mathematical epistemology than it really is. Other passages widen the gap between the two. In *Physics* B.2 Aristotle undertakes to explain how the mathematician differs from the physicist:

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[4] *Metaphysics*, M.3.1077b17—30, 1078a2—9, 21—26.

[5] *Ibid.*, K.1.1059b10—12. Compare B.2.997b35—998a6.

> Physical bodies have planes, solids, lengths, and points which the mathematician investigates . . . It would be absurd if the physicist were supposed to know what the sun or moon is but nothing about their essential properties, since physicists clearly do discuss the shape of the sun and moon and whether or not the earth or the cosmos is spherical. The mathematician too is concerned with these [properties] but not as limits of physical bodies. Nor does he investigate their properties as belonging to such bodies. Therefore he separates them; for in thought they are separable from change, and it makes no difference; nor does separation produce falsehood. Those who speak of ideas do the same thing without being aware of it; they separate the objects of physics, which are less separable than mathematical objects. This would become obvious if someone tried to give definitions of these [objects] and properties. For the even, the odd, the straight, the curved, and also number, line, figure would be without change; but flesh, bone, and man would not. These latter are defined like snub nose and not like the curved[6].

In this passage the idea of separation plays a much more important role than in *Metaphysics* M.3. The mathematician's separation of solids from the bodies which have them is something like the Platonists' separation of the forms. The solids, of course, cannot really exist separate from physical bodies, but they are somehow adapted to being considered separately. There seems to be a significant difference between separating mathematical objects from physical bodies and treating physical bodies as mathematical objects. I take the former to be essential to mathematics as Aristotle conceived it, and I shall try in what follows to make clear what separating amounts to.

Aristotle refers fairly frequently to mathematical objects as abstractions (ἐξ ἀφαιρέσεως, ἐν ἀφαιρέσει, δι' ἀφαιρέσεως)[7], but he does not explain what these expressions mean. The Greek verb, 'abstract,' (ἀφαιρεῖν) means 'take away' in a number of senses. I shall cite four passages where Aristotle uses this verb in a way relevant to mathematical abstractions[8]. In the *Posterior Analytics*,

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[6] *Physics*, B.2.193^{b}24—194^{a}7.

[7] See H. Bonitz, *Index Aristotelicus* (Berlin, 1870), 126^{b}21—26.

[8] For a fuller account of the relevant Aristotelian vocabulary, see M. D. Philippe, "Ἀφαίρεσις, Πρόσθεσις, Χωρίζειν dans la philosophie d' Aristote", *Revue Thomiste*, XLVIII (1948), 461—479.

he discusses the problem of determining the proper subject of some attribute, e. g., having angles equal to two rights. The method is to take away things until one finds the primary subject to which the attribute belongs. Aristotle's example is the bronze isosceles triangle with angles equal to two rights. From this the bronze and the isosceles must be taken away[9]. In Z.11 of the *Metaphysics* Aristotle speaks of taking away in thought the bronze from the bronze circle[10], disparaging the Platonists for taking away matter[11]. Earlier in Z, in discussing the question whether matter is substance, Aristotle speaks of taking away length, breadth, depth, and leaving matter[12].

In all of these passages, abstracting involves eliminating something from consideration. This is not a matter of collecting particulars and somehow arriving at a general idea[13], although abstraction is facilitated by seeing a number of different individuals. In the above-mentioned passage where Aristotle speaks of taking the bronze away from the bronze circle he says that this procedure would be hard if we never saw any non-bronze circles. Only once when he speaks of abstractions does Aristotle seem to imply that abstracting is a positive procedure and not just a matter of eliminating things from consideration. In the *Posterior Analytics* he says that abstractions are made known by induction[14]. However, the context makes it very likely that Aristotle is speaking of mathematical truths, i. e., assertions about abstractions, rather than of abstractions themselves. The commentators on this passage give

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[9] *Posterior Analytics*, A.5.74a33—74b1.

[10] *Metaphysics*, Z.11.1036a34—b3.

[11] *Ibid.*, 1036b22—23.

[12] *Ibid.*, Z.3.1029a16—19. Just prior to this passage at 1029a11 Aristotle uses περιαιρεῖν in the relevant sense of take away. Compare the use of this verb in *Categories* 7a31—b9.

[13] This notion of abstraction is the one which becomes crucial in British empiricism. See, for example, J. Locke, *An Essay Concerning Human Understanding*, II.xi.9: "The mind makes the particular ideas received from particular objects to become general; which is done by considering them as they are in the mind such appearances, separate from all other existences and the circumstances of real experience, as time, place, or any other concomitant *ideas*. This is called ABSTRACTION, whereby *ideas* taken from particular beings become general representatives of all of the same kind."

[14] *Posterior Analytics*, A.18.81b2—5. Charles Kahn has pointed out to me that τὰ ἐξ ἀφαιρέσεως λεγόμενα could be rendered "things asserted as a result of abstracting." Such a rendering would, of course, add support to the interpretation of the passage given above.

propositions as examples of abstractions[15]. Aristotle's point seems to be that the student is led to believe mathematical axioms by being shown that they hold in a number of particular cases.

Separating seems to be a correlate of abstracting. Abstracting from an object A gives rise to an object B lacking certain things belonging to A. Considering B is separating B. In the sequel I will refer to mathematical objects as abstractions, although it would be more accurate to call them separations or, to use a literal translation, "from-abstractions." We have already seen that Aristotle interprets mathematics as involving the separation of what is not separate or separable. It is easy to conclude that for Aristotle mathematical objects exist only in the mind of the mathematician and not independently of him. Such a conclusion involves failure to distinguish two kinds of separating. One kind is instanced in Plato's theory of forms and involves notions corresponding to nothing in reality. The other kind is instanced in mathematics and affords access to features of reality which are inaccessible in any other way[16]. The mathematician ignores certain features of the sensible world. The result, however, is not falsification, but knowledge, of the world. Mathematics is applicable to reality whereas the study of forms advocated by Plato has no application whatsoever. In the sequel I will argue that Aristotle explains this applicability by considering mathematical objects to underlie physical reality.

In places where Aristotle uses the verb 'abstract' in the relevant sense he speaks indifferently of taking away matter—e.g., bronze—and taking away properties—e.g., isosceles. There is, of course, nothing wrong with speaking in these two ways, but each of them if emphasized gives rise to a different notion of mathematical object. If abstraction is primarily thought of as eliminating properties, one will think of mathematical objects as physical objects looked at as if they did not have certain properties. On the other hand, if one thinks of abstraction primarily as eliminating matter,

[15] Themistius, *Analyticorum Posteriorum Paraphrasis*, ed. M. Wallies (Berlin, 1900), 31.4—32.2; Johannes Philoponus, *In Aristotelis Analytica Posteriora Commentaria*, ed. M. Wallies (Berlin, 1899), 215.15—24.

[16] Compare M. D. Philippe, "Ἀφαίρεσις. . .," 476—7: "[L'abstraction] ne crée pas [les êtres mathématiques], ne les invente pas, au sens fort, mais elle leur donne leur propre mode d'être. Elle les extrait pour ainsi dire du monde physique où ils se trouvaient comme cachés et enveloppés; elle les libère. L'abstraction joue, par rapport au monde mathématique, le rôle de l'experience par rapport au monde des autres sciences."

one will think of mathematical objects as properties like roundness and triangularity.

It is, I think, fair to say that the second view is more commonly attributed to Aristotle than the first. Both Philoponus and Simplicius make this attribution in interpreting the passage from the *Physics* quoted above: "The mathematician deals with figures and their properties, thinking these things to be embodied in no matter whatsoever; rather he studies the figures and their properties, separating them mentally from all matter[17]." "The mathematician differs from the physicist first because the physicist speaks not only about the properties of physical bodies but also about the matter; the mathematician is not concerned with matter[18]." All the commentators, moreover, use the contrast between the snub and the curved, referred to at the end of the passage, as an illustration of the difference between the natural scientist and the mathematician. In defining the snub, one must include reference to the matter in which it inheres—namely, a nose; but the curved can be defined and understood independently of matter.

The contrast between the properties of a thing and its matter is, of course, fundamental for Aristotle. In general he seems to treat properties as universals common to a number of individual things. If properties are universals, the mathematician who studies them will be studying universals. The doctrine that mathematical objects are properties fits, therefore, with Aristotle's account of demonstrative science in the *Posterior Analytics*. For underlying this account is the assumption that the basis of all scientific reasoning is the so-called categorical syllogism. But the categorical syllogism consists of sentences which, for Aristotle, are analyzable into terms. And these terms stand for universals:

> Therefore there do not have to be forms or some one apart from the many if there is to be proof, but there does have to be some one truly said of many. For if there were none, there would be no universal[19].

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[17] Johannes Philoponus, *In Aristotelis Physicorum Libros Tres Priores Commentaria*, ed. H. Vitelli (Berlin, 1887), 219.28—31.

[18] Simplicius, *In Aristotelis Physicorum Libros Quattuor Priores Commentaria*, ed. H. Diels (Berlin, 1882), 290.27—29. Compare Alexander of Aphrodisias (?), *In Aristotelis Metaphysica Commentaria*, ed. M. Hayduck (Berlin, 1891), 739.17—18: ὄντα γάρ εἰσιν, οὐκ ὡς ἔνυλα δέ, ἀλλ' ὡς ἄυλα καὶ εἴδη.

[19] *Posterior Analytics*, A.11.77^{a}5—7.

There are, then, good grounds for thinking Aristotle's view to have been that mathematical objects are universals separated in thought from matter. However, there are also grounds for doubt. First of all, universals do not have exactitude of the kind which Aristotle attributes to mathematical objects. Circularity does not touch straightness in a point or in any other way. For Aristotle, moreover, universals are not fully real. To suppose them to be the objects of mathematics would be incompatible with the assumption "that the truth of knowledge keeps pace with the actuality of its object". Of course, no supposition about mathematical objects will completely satisfy this assumption, given Aristotle's beliefs about the ontological primacy, but mathematical inadequacy, of sensible substances. One would, however, expect Aristotle to make at least some attempt to construe mathematical objects in the manner of sensible substances. This expectation is only strengthened by Greek mathematics itself, which is quite different from the study of universals. Its character is thoroughly geometric. The core of its reasoning is what we now call spatial intuition. Consequently, the objects in terms of which mathematical argument proceeds are intuitively perceived or imagined spatial objects, points, lines, plane figures, solids. Even numbers (positive integers) are represented by lines or points and thought of as collections of units subject to combinatorial manipulation. Finally, if Aristotle thought of mathematical objects as universals separated from matter, it is difficult to see how he could distinguish legitimate mathematical separation from illegitimate separation of Platonic forms.

Some of the difficulties involved in treating mathematical objects as universals are eliminated or lessened by treating them as particular properties, like the "certain white" of the *Categories*[20]. Such properties are perhaps more like sensible substances although they seem to be no more "exact" than universals. Moreover, in the *Metaphysics* Aristotle himself mentions a question which would make no sense if the objects of mathematics were matterless properties: "What is the matter of mathematical objects[21]?" Elsewhere he says explicitly that mathematical objects have intelligible matter[22]. And once he describes the process of abstraction as the elimination of sensible properties:

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[20] *Categories*, 2.1a27.
[21] *Metaphysics*, K.1.1059b14—16.
[22] *Ibid.*, Z.10.1036a9—12, Z.11.1037a2—5.

> The mathematician theorizes about abstractions, for he theorizes having removed all sensibles such as weight and lightness, hardness and its opposite, heat and cold, and the other sensible opposites. He leaves only the quantitative and continuous in one, two, or three [dimensions] and the properties of these as quantitative and continuous[23].

If the passage from *Metaphysics* M.3, quoted at the beginning of this paper, is interpreted in the light of the above, more sense can be made of it. To say that the mathematician studies a man as solid is not to say that he studies a man at all. Rather, it is to say that he studies what is quantitative and continuous in three dimensions. And the mathematician comes to understand the quantitative and continuous by abstracting—i.e., ignoring—all the sensible properties of some sensible substance such as a man. There is, then, at least an initial plausibility in supposing Aristotle to have entertained a conception of mathematical objects, not as matterless properties, but as substance-like individuals with a special matter—intelligible matter. I want now to explain this conception in more detail, to introduce more evidence for Aristotle's holding of it, and to show how it is related to the conception of mathematical objects as properties.

The first four categories discussed by Aristotle in the *Categories* are substance, quantity, relation, and quality. He himself gives no explanation for this order, and probably no deep significance attaches to it. However, all the commentators begin their discussion of chapter 6 of the *Categories*, the chapter on quantity, by attempting to explain why it immediately succeeds the chapter on substance. Many of the reasons advanced are trivial (e.g., "Substance can be primary or secondary but primary and secondary belong to quantity[24]."). But one reason mentioned by all the commentators is quite interesting; Philoponus states it thus:

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[23] *Ibid.*, K.3.1061a28—35. There seems to be general agreement that K.1—8 is genuinely Aristotelian in content although perhaps written by a student on the basis of lectures by Aristotle. (See e.g., *Aristotle's Metaphysics*, ed. with intro. and comm. by W. D. Ross [Oxford, 1924], I, p. xxvi.) The greater attention paid to mathematics in K than in the parallel B is perhaps explained by Jaeger's assumption of K's temporal priority. (See *Aristotle* [Oxford, 1934], pp. 208—21.) As Aristotle's thought evolved away from Platonism he may have considered questions of mathematical ontology less interesting. I have not found satisfactory evidence of changes in Aristotle's conception of mathematical objects.

[24] Elias, *In Porphyrii Isagogen et Aristotelis Categorias Commentaria*, ed. A. Busse (Berlin, 1900), 185.32—33.

> Prime matter, which is without body, form, or figure before it is filled out, receives the three dimensions and becomes three-dimensional. This Aristotle calls the second substratum, since thus it receives quality and produces the elements[25].

The term 'second substratum' is not found in the extant works of Aristotle. Probably it should not be ascribed to him. Nor can one assume that Aristotle reasoned this way in organizing the *Categories*. Yet the basic idea of the reasoning does seem Aristotelian; for Aristotle thinks of quantity in the way this reasoning suggests, as chapter 6 itself shows. There Aristotle does not, as one would expect,

> list or attempt to classify quantitative properties (like the property of being a foot long) or corresponding predicates (like 'a foot long'). Instead he lists and groups owners of quantitative properties: lines, surfaces, solids, numbers (aggregates), time periods, places, utterances[26].

This difference between the discussion of quantity and the other non-substantial categories does not seem to be accidental. The quantitative and continuous, which Aristotle says remain after the sensible opposites have been removed, seem to be identical with these first three quantities: lines, surfaces, solids. And even in a passage where he affirms the difference between quantity and substance, the three dimensions are treated as more fundamental than properties:

> Other things are properties, actions, and powers of bodies, but length, breadth, and depth are certain quantities and not substances. For quantity is not substance, which is rather that to which these things primarily belong. But, when length, breadth, and depth are taken away, we see nothing remaining, unless what is bounded by these is something[27].

We find, then, in Aristotle the notion that if one abstracts properties in the proper order one is left with the idea of an object having only length, breadth, and depth, the continuous and

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25 Johannes Philoponus, *In Aristotelis Categorias Commentarium*, ed. A. Busse (Berlin, 1898), 83.14—17.

26 *Aristotle's Categories and De Interpretatione*, tr. with notes by J. Ackrill (Oxford, 1963), p. 91.

27 *Metaphysics*, Z.3.1029a12—18.

quantitative in three dimensions, the solid. From this idea by abstraction one may obtain the idea of length and breadth, the continuous and quantitative in two dimensions, the plane; and by further abstraction the idea of length alone, the continuous and quantitative in one dimension, the line. The point is not a quantity for Aristotle because it cannot be measured. It is, however, a basic geometric object presupposed in many geometric constructions. Aristotle's notion of how we grasp the idea of a point is not completely clear. Sometimes he says the point is what has position and is indivisible[28]; at other times he characterizes it as the limit or division of a line[29]. The latter suggestion seems to fit better with the process by which we come to understand solids, planes, and lines. Perhaps the former should be thought of as the logically correct definition of point rather than the description of our ordinary conception of it[30].

Regardless of how he treats points, Aristotle seems to have the idea of the purely dimensional underlying other properties. In part this is the idea of the three-dimensional underlying sensible properties in the physical world. But for Aristotle there is little if any difference between this idea and that of the one-, two-, or three-dimensional underlying geometric properties, which he calls intelligible matter. The phrase 'intelligible matter' is found in three passages in the *Metaphysics*. In two of them it is introduced in connection with the problem of explaining in what sense a semicircle is a part of a particular mathematical circle[31]. The author of the commentary on the *Metaphysics* which is attributed to Alexander of Aphrodisias refers to the intelligible matter in these passages as extension[32], and he is clearly right. For it is the extendedness of geometric objects, their continuity in one, two, or three dimensions, which makes them divisible. The third passage causes some difficulty, however. Aristotle says: "Some matter is intelligible, some sensible; and part of a definition is always matter, part actuality. For example, a circle is a plane figure[33]." "Alexander" says that

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[28] E.g., at *Metaphysics*, Δ.6.1016^{b}24—31.
[29] E.g., at *Metaphysics*, K.2.1060^{b}12—17.
[30] Compare H. Apostle, *Aristotle's Philosophy of Mathematics* (Chicago, 1952), p. 100.
[31] See note 20.
[32] Alexander (?), *In Metaphysica*, 510.3—5, 515.26—28.
[33] *Metaphysics*, H.6.1045^{a}33—35.

'intelligible matter' here means 'genus'—genera being "analogous" to matter. According to him, figure is an example of intelligible matter[34]. W. D. Ross accepts this interpretation; but in his view plane figure is an example of intelligible matter and Aristotle fails to give the formal element in defining circle[35], namely, "contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another[36]."

If we look at this passage in the light of the previous discussion, it becomes clear that plane, i.e., the continuous and quantitative in two dimensions, is the element of matter in the definition of circle and that figure stands for the formal element. This interpretation is confirmed in the discussion of genus in *Metaphysics* Δ.28: "The plane is genus of plane figures, the solid of solid; for each of the figures is such and such a plane or solid, and this is the substratum of the differentiae[37]." Thus it becomes necessary to distinguish two kinds of geometric object in Aristotle. First, there are the basic objects: points, lines, planes, solids. The last three are conceived of as indeterminate extension and, therefore, as matter on which geometric properties are imposed. The imposition of these properties produces the ordinary geometric figures, straight or curved lines, triangles, cubes, etc. The definition of such a figure will include both the form, the properties imposed, and the matter; but in the definition this matter will also play the role of genus. A circle is a plane figure.

The distinction between the quantitative substratum and the geometric properties imposed on it has an important function in Aristotle's account of the first principles or elements of a demonstrative science in the *Posterior Analytics*. Aristotle says that there are three kinds of element: (1) the common axioms; (2) the genus, the things whose existence and meaning are assumed; (3) the properties, whose meaning only is assumed[38]. As examples of the genus he gives the unit or units, points, lines, magnitude; as examples of properties, odd, even, square, cube, straight, triangle,

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34 Alexander (?), *In Metaphysica*, 562.14—17.
35 *Aristotle's Metaphysics*, II, p. 238.
36 Euclid, *Elements*, Bk. I, def. 15. Translation by T. Heath, *The Thirteen Books of Euclid's Elements* (Cambridge, England, 1925), I, 153.
37 *Metaphysics*, Δ.28.1024a36—b4.
38 *Posterior Analytics*, A.10.76a31—36, 76b11—16.

incommensurable, inclination, deflection[39]. If allowance is made for the difficulty about points the distinction between genus and property is seen to be exactly the distinction between intelligible matter and form in the *Metaphysics*. For in Aristotle 'magnitude' is just a general term referring to lines, planes, and solids[40]. These examples also show how different Aristotle's conception of geometry is from the one represented by Euclid's *Elements*. In both, the meaning of all terms used is supposed known, and common axioms are assumed. But the postulates of Euclid do not give the genus being discussed, rather, three simple constructions and two assumptions on the basis of which theorems and other constructions are justified. Aristotle looks at geometry quite differently. The point of departure in, say, plane geometry is extensionality in two dimensions (planes) with the concomitants of extensionality in one dimension (lines) and positional dimensionlessness (points). One must not only understand what these things are; one must also suppose them to exist. For, unlike the hot and the cold, the existence of these things is not obvious and cannot be just an implicit assumption[41]. Once these things are understood and posited, geometry proceeds by examining properties like straightness or triangularity which they possess. Because the geometer is primarily concerned with properties, Aristotle often speaks of him as separating properties. However, although the geometer does consider geometric properties separate from the physical things in which they inhere, geometric objects are none the less compounds of these properties and intelligible matter.

If Aristotle held the conception of geometric objects which I have developed here, it is easy to see how their exactitude is explained. For by abstraction one eliminates all sensible characteristics and arrives at the idea of pure extension. Pure extension does not seem to be sensible in the way that triangularity is, nor is it completely undifferentiated or purely potential in the way that prime matter seems to be. We cannot see a thing as just extended but only as extended so and so much with a certain shape. Simple extendedness we must grasp rationally. Geometric properties are imposed on this intelligible matter, but these properties are not the approximate properties of sensible substances precisely because they are imposed upon intelligible matter. The resultant objects are still intelligible

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39 *Ibid.*, 76ᵃ34—36, 76ᵇ3—11.

40 See, for example, *Metaphysics*, Δ.13.1020ᵃ7—14, or *De Caelo*, A.1.268ᵃ7—8.

41 *Posterior Analytics*, A.10.76ᵇ16—19.

rather than sensible. In postulating such objects, the mathematician separates what is not separable. Intelligible matter, even when endowed with geometric properties, is no more capable of existing separately than are sensible properties. Such separation is not harmful, however, because the separated objects are rationally comprehensible and closely connected with the real world. According to Aristotle, at any rate, the idea of a man separated from flesh and bones is an incomprehensible fabrication.

In the final part of this paper I want to bring together the things I have said about Aristotle's conception of geometric objects. Aristotle starts from the Platonic notion of geometry as the study of forms and from the intuitive character of Greek geometry. The former leads him to the idea of geometry as the study of universals, an idea most fully embodied in the *Posterior Analytics*. The latter leads him to the idea of geometry as the study of objects which result from the combination of geometric properties and intelligible matter. The problem is to reconcile these two ideas.

Some of the neo-Platonists, who tried to reconcile everything in Plato and Aristotle, attribute to Aristotle a distinction between two kinds of reasoning about two kinds of objects corresponding to the upper half of the divided line in Plato's *Republic*. Proclus speaks of reasoning about forms perfectly embodied in the φαντασία and, secondly, of reasoning about disembodied forms—forms having, therefore, no spatial properties[42]. Proclus' suggestion is that Aristotle made this distinction in terms of active and passive intellect[43], a suggestion obviously without basis in what Aristotle says. The most significant feature of Proclus' distinction is perhaps that it makes particular geometric objects mental objects—objects having their subsistence in the φαντασία. He invokes Aristotle on this point too by referring to an alleged Aristotelian distinction between two kinds of matter, "one of things correlated to sense, the other of imagined things[44]." By the latter Proclus presumably means intelligible matter, but there is no good reason to suppose that Aristotle thought of intelligible matter as mental or imagined. Indeed, although Aristotle explicitly recognizes the role of imagina-

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42 Proclus, *In Primum Euclidis Elementorum Librum Commentarii*, ed. G. Friedlein (Leipzig, 1873), 48.1—56.22. A similar doctrine is attributed to Aristotle by Simplicius in his commentary on *De Anima*—*In Libros Aristotelis De Anima Commentaria*, ed. M. Hayduck (Berlin, 1882). See, for example, 233.7—35.

43 Proclus, *In I. Euclidis Elementorum*, 52.3—4.

44 *Ibid.*, 51.13—17.

tion in geometric thinking[45] and although he calls the mathematician's act of separation mental, he always connects mathematical objects directly with the sensible world. "Physical bodies have planes, solids, lengths, and points which the mathematician investigates." "There could be assertions and proofs about sensible magnitudes not as sensible."

Not only does Aristotle seem unwilling to make the objects of mathematics mental but also he never espouses a division of scientific reasoning into two kinds along the lines suggested by Proclus. The mathematics described by Aristotle in the *Posterior Analytics* is for him basically that of the ordinary geometer. Nor does the discrepancy between the two kinds of geometric object seem to be a special case of the general difficulty which Aristotle mentions and tries to handle in the *Metaphysics:* How can knowledge be of universals if what is most real is the particular[46]? For Aristotle treats this difficulty only as a general problem arising from the attempt to combine the theory of knowledge of the *Posterior Analytics* with the ontology of the *Metaphysics*. He does not treat it as a problem relating to any particular mathematical or physical science.

I am inclined to think we must credit Aristotle with a much more subtle and reasonable position than the one attributed to him by Proclus. There is clearly some distinction to be made between knowledge of universals or conceptual knowledge and intuitive knowledge[47]. Both are a part of geometry, but the mistake of Proclus and others is to assume that a sharp distinction can be made between the two. In the words of P. Bernays:

> The sharp separation of intuition and concept . . . does not appear on closer examination to be justified. In considering geometrical thinking in particular it is difficult to distinguish clearly the share of intuition from that of conceptuality, since we find here a formation of concepts guided so to speak by intuition, which in the sharpness of its intentions goes beyond what is in a proper sense intuitively evident, but

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[45] *De Anima*, Γ.8.432a3—10.

[46] *Metaphysics*, B.6.1003a5—17, K.2.1060b19—23, M.10.1087a10—25.

[47] In the last passage cited in the previous footnote Aristotle makes an analogous distinction between knowledge of the particular and knowledge of the universal in terms of a distinction between actual and potential knowledge. I have been unable to apply this distinction to geometry in a satisfactory way.

which separated from intuition has not its proper content[48].

The intimate connection of concept and intuition underlies, I suggest, Aristotle's account of geometry and geometric objects. It is illustrated most clearly in Aristotle's treatment of extension as a kind of underlying stuff and as a very abstract notion, the genus of mathematical objects. A modern analogue of Aristotelian extension would be space, which is formally nothing but a class of elements (points) related in certain ways but which is also in some sense an object of perception or intuition.

Aristotle's account of geometric objects would seem, then, to be something like the following. In his reasoning the geometer deals directly with the particular geometric objects which I have been describing. These objects, though not real in the sense in which sensible substances are, are intimately connected with sensible reality and in a certain sense underlie it. However, out of ordinary geometric reasoning arises a universal knowledge, e. g.,the knowledge that any triangle has interior angles equal to two rights. Universal knowledge is conceptual and can be formulated syllogistically. However, it has no object over and above the objects of ordinary geometric reasoning, and in fact conceptual syllogistic reasoning is only a reformulation of ordinary reasoning[49]. Thus there is only one kind of geometric object although there are two ways of reasoning about it, one apparently more abstract than the other[50].

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[48] P. Bernays, "Comments on Ludwig Wittgenstein's *Remarks on the Foundations of Mathematics*," reprinted in *Philosophy of Mathematics*, p. 518.

[49] Aristotle was, of course, mistaken in believing geometric reasoning could be represented syllogistically, but in this context his mistake can be treated as one of detail not of principle.

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